**Redshifts and Distances**

Spacetime and Geometry : An Introduction to General Relativity – by Sean M Carroll

By George Keeling, created 24 Feb 2021

# Summary

In section 8.5 we are looking at redshifts and distances. The latter are more complicated than you might think! We start in an FLRW universe with metric

|  |  |
| --- | --- |
|  | (1) |

We then found a rank 2 Killing tensor

|  |  |
| --- | --- |
|  | (2) |

where is the 4-velocity of (all) comoving observers and . We proved it is a Killing tensor in 'Commentary 8.5 Killing Tensor in FLRW spacetime'.

Summarising the business:

* We use to show that interstellar gas cools in an expanding universe (7).
* Then we use it to show that as the universe expands the observed frequency of a photon will decrease (16).
* Cosmologist measure the redshift, , of distant objects and we show how this tells us the scale factor when the object emitted the light being measured (20).
* I then looked up some galactic distances and redshifts to compare them. Extrapolating the graph indicates the size of the visible universe is ~15 Gly (21).
* Carroll then seems to ramble a bit and goes back to a non-relativistic redshift (36). The history of Doppler and Hubble, which he does not mention, is quite interesting.
* Getting back on track Carrol shows that the nearby universe can be thought of as flat and we are able to use proper distance (42) meaningfully and derive Hubble's law (46). I think he is overcomplicated.
* Next step other distances!

# The Business

Extending our rule about Killing vectors a bit (Carroll 3.175) we have is conserved along geodesics. is the four-momentum of some (any) particle. Mass is conserved so that means if its four velocity is

|  |  |
| --- | --- |
|  | (3) |

is conserved for the particle along geodesics. Call the conserved quantity and

|  |  |
| --- | --- |
|  | (4) |

is constant for the particle along geodesics.

## Massive particles (interstellar gas)

For massive particles and so

|  |  |
| --- | --- |
|  | (5) |

where we introduced where is very like the three-velocity of the particle. Since we are thinking of a particle on a geodesic, it is in free-fall. We also have so (4) becomes

|  |  |
| --- | --- |
|  | (6) |
|  | (7) |

which is Carroll's 8.101 and shows that particles in free fall "slow down" as the scale factor increases. I suppose the quotes are there because is not quite your usual three velocity which is but the quantity . Also we have not used the true fact that or Carroll's equation 8.100 which was also true. Both seem irrelevant.

As Carroll says free moving particles, such as interstellar gas, slow down. or cool, as the universe expands. Is that surprising?

## Photons

For photons the line element equation is

|  |  |
| --- | --- |
|  | (8) |
|  | (9) |

where is the four-velocity of the photon. So that's the easy place where Carroll's comes from.

Stick that into (4) and we get

|  |  |
| --- | --- |
|  | (10) |
|  | (11) |

Then back in section 3.4 we had equations 3.62 and 3.63: The four-momentum of a photon is

|  |  |
| --- | --- |
|  | (12) |

Any observer with four-velocity measures energy of a photon with four momentum to be

|  |  |
| --- | --- |
|  | (13) |

and we proved that in 'Commentary 3.4 Particle energy'. Of course we have

where is the photon frequency and we are using .

So our is the four-momentum and our comoving observer with four-velocity will measure the photon frequency as

|  |  |
| --- | --- |
|  | (14) |

The latter is easy to show by starting with raising and lowering indices, then contracting. So combining that with (11) we get

|  |  |
| --- | --- |
|  | (15) |

We think that so , which might be interesting. Whatever we certainly get that as the universe expands the observed frequency of a photon will decrease:

|  |  |
| --- | --- |
|  | (16) |

If the universe was contracting the observed frequency would increase. But we know the universe is expanding because Hubble measured it. Turning that into wavelengths of light we get

|  |  |
| --- | --- |
|  | (17) |

and the fractional change in wavelength , the **redshift** of the emitting source which is what cosmologists like, is

|  |  |
| --- | --- |
|  | (18) |

and if the observation takes place today ()

|  |  |
| --- | --- |
|  | (19) |

which gets us to Carroll's 8.105

|  |  |
| --- | --- |
|  | (20) |
| So the redshift tells us the scale factor where the photon was when it was emitted (the emission event). |  |

The table below and accompanying graph show a selection of galaxies, their distances in billion light years, redshift and scale factor when they emitted a photon that we saw in the current epoch (today years). It starts with our close neighbour Messier 87 and includes quasars, which are black holes at the centre of a galaxy, and one super cluster of galaxies.

I suppose redshift is measured from emission lines of some element. I am not sure how the distances are measured. The distance and redshift numbers are presented as they were on Wikipedia - for each galaxy they seem to usually be to different accuracies so might be measured independently. If on the other hand the distance is calculated from the redshift then the correlation is spurious. I was surprised at the apparently linear decrease of the scale factor. What happened to the accelerating expansion of the universe?

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Galaxy / quasar | Distance Gly | Redshift | Scale factor | Scale factor × 10 |
| [Messier 87](https://en.wikipedia.org/wiki/Messier_87) (size ) |  | 0.0043 | 1.00 | 9.96 |
| [Saraswati Supercluster](https://en.wikipedia.org/wiki/Saraswati_Supercluster) | 4.01 | 0.28 | 0.78 | 7.81 |
| [RX J1131-1231](https://en.wikipedia.org/wiki/RX_J1131-1231) | 6.05 | 0.658 | 0.60 | 6.03 |
| [Einstein Cross](https://en.wikipedia.org/wiki/Einstein_Cross) | 8.0 | 1.695 | 0.37 | 3.71 |
| [Twin Quasar](https://en.wikipedia.org/wiki/Twin_Quasar) | 8.7 | 1.413 | 0.41 | 4.14 |
| [Cloverleaf quasar](https://en.wikipedia.org/wiki/Cloverleaf_quasar) | 11 | 2.56 | 0.28 | 2.81 |
| [LBG-2377](https://en.wikipedia.org/wiki/LBG-2377) | 11.4 | 3.03 | 0.25 | 2.48 |
| [Cosmos Redshift 7](https://en.wikipedia.org/wiki/Cosmos_Redshift_7) | 12.9 | 6.604 | 0.13 | 1.32 |

1) Comoving distance

|  |  |
| --- | --- |
|  | (21) |

## Instantaneous physical distance

### Small compared to Hubble radius

Carroll then says that this redshift can be thought of as not cosmological over distances small compared to the Hubble length (which he now calls radius, discarding the jargon established in section 8.3) which is . In Commentary 3.5 Expanding Universe Revisited #3 we mapped a Minkowski space time onto and FLRW spacetime with an approximation and found that determined the limit to which the approximation worked. So that explains why we can treat redshift as not cosmological over such distances.

### Small compared to radius of spatial curvature

Carroll then says that, additionally, distances must be small compared with the 'radius of spatial curvature' . From the (1) the metric equation we immediately remember that must be dimensionless, so and has the correct dimensions for Carroll's assertion. On a 2-sphere the curvature scalar is given by

|  |  |
| --- | --- |
|  | (22) |

where is the radius of the two sphere. We showed that in 'Commentary 8 Curvatures 2D'. As increases and the two sphere gets flatter and flatter. If distances on the two sphere are small compared to it appears flat. (Like the Netherlands on the surface of the earth!) So Carroll is saying that is like . Why?

We're thinking about hypersurfaces of constant which have a metric (like Carroll's 8.29)

|  |  |
| --- | --- |
|  | (23) |

if then and that metric is almost Minkowski in spherical coordinates - flat. So now we know why distances must be small compared with and we have an inkling of why he calls the 'radius of spatial curvature'. It doesn't really matter whether the universe is positively or negatively curved. When it's negative (open) the radius term does not make much sense. Carroll has fallen in love with the word radius here.

### Formula for velocity in terms of redshift *z* relativistic

Now we have established that we can think of redshift as Doppler redshift in a flat universe and Carroll says that, "consequently, astronomers often think of the redshift in terms of "velocity" . Where does that come from? Why would astronomers think that?

If we interpret the redshift as relativistic Doppler redshift then we have

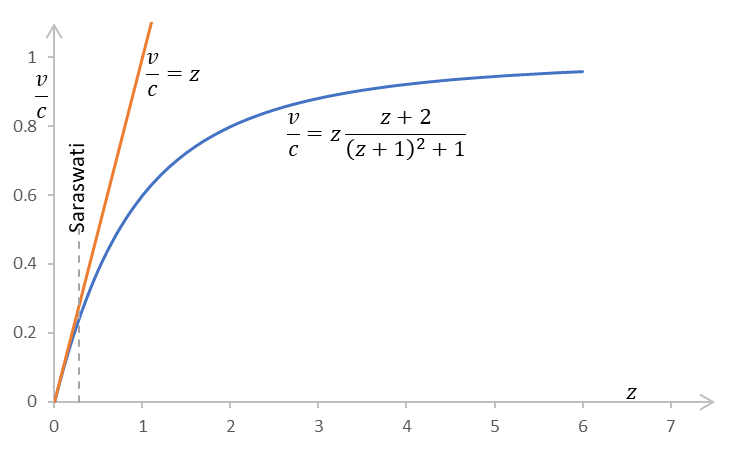
|  |  |
| --- | --- |
|  | (24) |

from (18) dropping the subscript on that gives us

|  |  |
| --- | --- |
|  | (25) |
|  | (26) |
|  | (27) |
|  | (28) |
|  | (29) |

reinstating the speed of light

|  |  |
| --- | --- |
|  | (30) |



Compare against the improved formula at (30) shows how wrong it is. It doesn't even work very well for Saraswati and gives a superluminal velocity for the Einstein Cross and the others.

### Formula for velocity in terms of redshift *z* non-relativistic

What about classical doppler redshift? Could that be where comes from? Yes!

The Austrian physicist Christian Doppler described the phenomenon in 1842 on a treatise on the colour of stars so presumably he was thinking of light waves and, from the formula, thought that the speed of light was relative to some aether. His formula is

|  |  |
| --- | --- |
|  | (31) |

where

is the frequency observed

is the frequency of emission

is the propagation speed of waves in the medium

is the speed of the observer relative to the medium (aether), added to if the observer is moving towards the emitter

is the speed of the emitter relative to the medium, added to if the emitter is moving away from the observer

In our case we imagine the observer at rest, , and the emitter is moving away, so that becomes

|  |  |
| --- | --- |
|  | (32) |

we also have the relationship so that becomes

|  |  |
| --- | --- |
|  | (33) |

Using (18) and dropping the subscript on again we find that

|  |  |
| --- | --- |
|  | (34) |
|  | (35) |

and arrive at the mysterious

|  |  |
| --- | --- |
|  | (36) |

|  |  |
| --- | --- |
| If we zoom in to the origin of our graph we get the picture on the right so we see that the formula is OK for and referring back to the graph of eight galaxies that is equivalent to a distance of less than 0.1 billion light years (Gly) which is a small fraction of the size of the visible universe which is about . |  |

Edwin Hubble (1889-1953) was the first person to identify galaxies outside the Milky Way. In 1922 Messier 87 (53 Mly) joined his list. Andromeda (2.5 Mly) and Triangulum (2.7 Mly) were also on it. I suppose that all the galaxies were within the 100 Mly limit so the formula would work well even thought Hubble must have known and believed about SR Doppler shift. He "remained doubtful about Lemaitre's interpretation for his entire life".

## Instantaneous physical distance #2

Having divested ourselves of any hint of relativity we now reinvoke it and go back to Carroll's 8.38 the FLRW metric

|  |  |
| --- | --- |
|  | (37) |

and reintroduce the substitution and solutions (Carroll 8.31, 8.33, 8.40)

|  |  |
| --- | --- |
|  | (38) |
|  | (39) |
|  | (40) |

so (37) becomes

|  |  |
| --- | --- |
|  | (41) |

which is Carroll's 8.106, a sort of hybrid FLRW metric. Now if we take an instantaneous () radial () slice we have

|  |  |
| --- | --- |
|  | (42) |

is the proper distance (distance measured at constant ) or, when integrated, the 'instantaneous physical distance' between where we are and where some nearby galaxy is. So

|  |  |
| --- | --- |
|  | (43) |

and the recession velocity (inferred from redshift and presumably using ) is

|  |  |
| --- | --- |
|  | (44) |

now just use (43) to get rid of the slightly mysterious , which was only ever used as a fleeting intermediary to get us from the (37) metric to the (1) metric, and we have

|  |  |
| --- | --- |
|  | (45) |

Evaluated today that is Hubble's law

|  |  |
| --- | --- |
|  | (46) |

We could have done that much simpler directly from (1). We have already said (23) that then and that spatial metric is almost Minkowski in spherical coordinates - flat. So (1) becomes

|  |  |
| --- | --- |
|  | (47) |

on our instantaneous radial measure that is

|  |  |
| --- | --- |
|  | (48) |
|  | (49) |

where is now the constant comoving distance to the galaxy which we could measure in meters and from there we immediately get

|  |  |
| --- | --- |
|  | (50) |

There was no need for Carroll's smoke and mirrors!

However in the next part on luminosity distance we continue to use the hybrid metric, so perhaps the exercise was useful.

**References**

My blog: <https://www.general-relativity.net/>

docx file: [Commentary 8.5 Redshifts and Distances.docx](https://drive.google.com/open?id=1ixUlVR8jUVDSPlsHU_ePMvn3jazaCB6U)

Excel: [Commentary 8.5 Redshifts and Distances.xlsx](https://drive.google.com/open?id=1GLk0zbqatn_mvJp8aOtTzpjqF2S4ADA_)

[Commentary 8.5 Killing Tensor in FLRW spacetime.pdf](https://drive.google.com/open?id=1eleWMbEL5LccVIrc_q5IdmDFpydTGZL_)

[Commentary 3.4 Particle energy.pdf](https://drive.google.com/open?id=1DdHVFoJBPsUe0NvOCxYs7PtV8yAMSQ0C)

[Commentary 3.5 Expanding Universe Revisited #3.pdf](https://drive.google.com/open?id=12GyuyimvKc-_Nn-305ULhhwGy8EaByO4)

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Doppler effect

<https://en.wikipedia.org/wiki/Doppler_effect>

Edwin Hubble

<https://en.wikipedia.org/wiki/Edwin_Hubble>